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STUDY PACKAGE

Subject : Mathematics

Topic : Applications of Derivatives

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Tangent & Normal

Teko Classes, Maths : Suhag R. Kariya (S. R. K. Sir), Bhopal Phone :0 903 903 7779, 0 98930 58881.

A. Derivative as rate of change

If the quantity y varies with respect to another quantity x satisfying some relation $y = f(x)$, then $f'(x)$ or $\frac{dy}{dx}$ represents rate of change of y with respect to x.

Example : The volume of a cube is increasing at rate of 7 cm³/sec. How fast is the surface area increasing when the length of an edge is 4 cm?

Solution. Let at some time t, the length of edge is x cm.

$$v = x^3 \Rightarrow \frac{dv}{dt} = 3x^2\frac{dx}{dt} \quad \left(\text{but } \frac{dv}{dt} = 7\right)$$

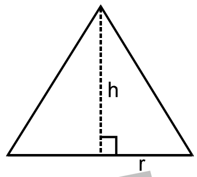
$$\Rightarrow \frac{dx}{dt} = \frac{7}{3x^2} \text{ cm/sec.}$$

Now $s = 6x^2$

$$\frac{ds}{dt} = 12x\frac{dx}{dt} \quad \Rightarrow \quad \frac{ds}{dt} = 12x.\frac{7}{3x^2} = \frac{28}{x}$$

when x = 4 cm $\quad \frac{ds}{dt} = 7$ cm²/sec.

Example : Sand is pouring from pipe at the rate of 12 cm³/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one - sixth of radius of base. How fast is the height of the sand cone increasing when height is 4 cm?



Solution. $v = \frac{1}{3}\pi r^2 h$

but $h = \frac{r}{6}$

$$\Rightarrow v = \frac{1}{3}\pi(6h)^2 h \quad \Rightarrow \quad v = 12\pi h^3$$

$$\frac{dv}{dt} = 36\pi h^2.\frac{dh}{dt}$$

when, $\frac{dv}{dt} = 12$ cm³/s and h = 4 cm

$$\frac{dh}{dt} = \frac{12}{36\pi.(4)^2} = \frac{1}{48\pi} \text{ cm/sec.}$$

Self practice problem :

1. Radius of a circle is increasing at rate of 3 cm/sec. Find the rate at which the area of circle is increasing at the instant when radius is 10 cm. **Ans.** 60π cm²/sec

2. A ladder of length 5 m is leaning against a wall. The bottom of ladder is being pulled along the ground away from wall at rate of 2cm/sec. How fast is the top part of ladder sliding on the wall when foot of ladder is 4 m away from wall. **Ans.** $\frac{8}{3}$ cm/sec

3. Water is dripping out of a conical funnel of semi-vertical angle 45° at rate of 2cm³/s. Find the rate at which slant height of water is decreasing when the height of water is $\sqrt{2}$ cm. **Ans.** $\frac{1}{\sqrt{2\pi}}$ cm/sec.

4. A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment. **Ans.** 140 ft/min.

B Equation of Tangent and Normal

$\left.\frac{dy}{dx}\right|_{(x_1, y_1)} = f'(x_1)$ denotes the slope of tangent at point (x_1, y_1) on the curve $y = f(x)$. Hence the equation of tangent at (x_1, y_1) is given by

$$(y - y_1) = f'(x_1)\,(x - x_1)$$

Also, since normal is a line perpendicular to tangent at (x_1, y_1) so its equation is given by

$$(y - y_1) = -\frac{1}{f'(x_1)}\,(x - x_1)$$

Example: Find equation of tangent to $y = e^x$ at x = 0.

Solution At x = 0 $\Rightarrow$ $y = e^0 = 1$

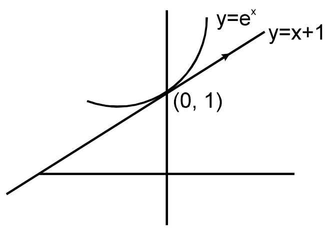
Hence point of tangent is (0, 1)

$$\frac{dy}{dx} = e^x \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=0} = 1$$

Hence equation of tangent is

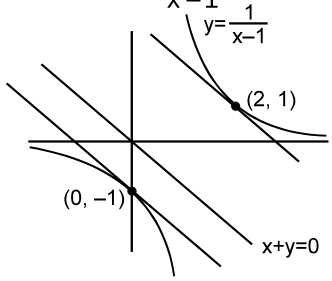
$1\,(x - 0) = (y - 1)$

$\Rightarrow \quad y = x + 1$



Example : Find the equation of all straight lines which are tangent to curve $y = \frac{1}{x-1}$ and which are parallel to the line $x + y = 0$.

Solution : Suppose the tangent is at (x_1, y_1) and it has slope -1.



$\Rightarrow \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = -1.$

$\Rightarrow -\frac{1}{(x_1-1)^2} = -1.$

$\Rightarrow x_1 = 0$ or 2

$\Rightarrow y_1 = -1$ or 1

Hence tangent at $(0, -1)$ and $(2, 1)$ are the required lines with equations

$-1(x - 0) = (y + 1)$ and $-1(x - 2) = (y - 1)$

$\Rightarrow x + y + 1 = 0$ and $y + x = 3$

Example : Find equation of normal to the curve $y = |x^2 - |x||$ at $x = -2$.

Solution In the neighborhood of $x = -2$, $y = x^2 + x$.

Hence the point of contact is $(-2, 2)$

$\frac{dy}{dx} = 2x + 1 \Rightarrow \left.\frac{dy}{dx}\right|_{x=-2} = -3.$

So the slope of normal at $(-2, 2)$ is $\frac{1}{3}$.

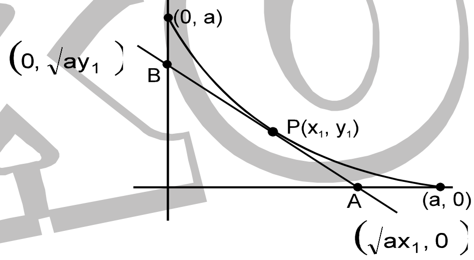
Hence equation of normal is

$\frac{1}{3}(x + 2) = y - 2. \Rightarrow 3y = x + 8.$

Example : Prove that sum of intercepts of the tangent at any point to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ on the coordinate is constant.

Solution : Let $P(x_1, y_1)$ be a variable point on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$

$\Rightarrow$ equation of tangent at point p is $-\frac{\sqrt{y_1}}{\sqrt{x_1}}(x - x_1) = (y - y_1)$



$\Rightarrow -\frac{x}{\sqrt{x_1}} + \sqrt{x_1} = \frac{y}{\sqrt{y_1}} - \sqrt{y_1}$

$\Rightarrow \frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \sqrt{x_1} + \sqrt{y_1}$

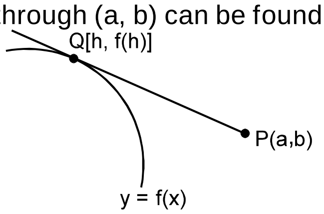
$\Rightarrow \frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \sqrt{a}$ $(\because \sqrt{x_1} + \sqrt{y_1} = \sqrt{a})$

Hence point A is $(\sqrt{ax_1}, 0)$ and coordinates of point B is $(0, \sqrt{ay_1})$. Sum of intercepts

$= \sqrt{a}(\sqrt{x_1} + \sqrt{y_1}) = \sqrt{a}.\sqrt{a} = a.$

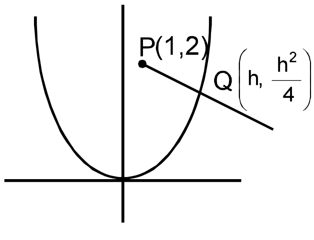
C. Tangent from an External Point

Given a point P(a, b) which does not lie on the curve $y = f(x)$, then the equation of possible tangents to the curve $y = f(x)$, passing through (a, b) can be found by solving for the point of contact Q.



Example : Find the equation of all possible normal to the parabola $x^2 = 4y$ drawn from point (1, 2).

Solution Let point Q be $\left(h, \frac{h^2}{4}\right)$



Now, m_{PQ} = slope of normal at Q.

Slope of normal $= -\left.\frac{dx}{dy}\right|_{x=h} = -\frac{2}{h}$

$\Rightarrow \frac{\frac{h^2}{4} - 2}{h - 1} = -\frac{2}{h}$

$\Rightarrow \frac{h^3}{4} - 2h = -2h + 2$

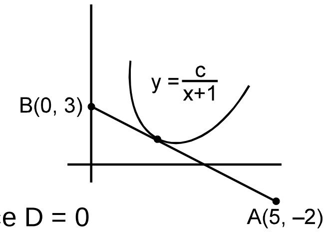
$\Rightarrow \quad h^3 = 8 \quad \Rightarrow \quad h = 2$

Hence coordinates of point Q is (2, 1) and so equation of required normal becomes x + y = 3.

Note : The equation gives only one real value of h, hence there is only one point of contact implying that only one real normal is possible from point (1, 2).

Example : Find value of c such that line joining points (0, 3) and (5, – 2) becomes tangent to curve $y = \dfrac{c}{x+1}$.

Solution. Equation of line joining A & B is x + y = 3



Solving this line and curve we get

$3 - x = \dfrac{c}{x+1} \quad \Rightarrow \quad x^2 - 2x + (c - 3) = 0 \quad$ ......(i)

For tangency, roots of this equation must be coincident. Hence D = 0

$\Rightarrow \quad 4 = 4(c - 3) \quad \Rightarrow \quad c = 4$

Note : If a line touches a curve then on solving the equation of line and tangent we get at least two repeated roots corresponding to point of contact.

Putting c = 4, equation (i) becomes

$x^2 - 2x + 1 = 0 \Rightarrow \quad x = 1$

Hence point of contact becomes (1, 2).

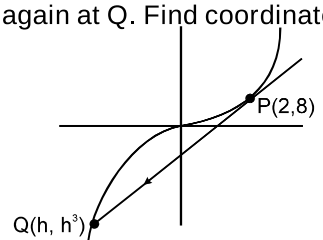
Example : Tangent at P(2, 8) on the curve $y = x^3$ meets the curve again at Q. Find coordinates of Q.

Solution. Equation of tangent at (2, 8) is

y = 12x – 16

Solving this with $y = x^3$

$x^3 - 12x + 16 = 0$



this cubic must give all points of intersection of line and curve $y = x^3$ i.e., point P and Q.

But, since line is tangent at P so x = 2 will be a repeated root of equation $x^3 - 12x + 16 = 0$ and another root will be x = h. Using theory of equations

sum of roots $\Rightarrow \quad 2 + 2 + h = 0 \quad \Rightarrow \quad h = -4$

Hence coordinates of Q are (– 4, – 64)

Self Practice Problems :

1. Find the slope of the normal to the curve $x = 1 - a \sin\theta$, $y = b\cos^2\theta$ at $\theta = \dfrac{\pi}{2}$. **Ans.** $-\dfrac{a}{2b}$

2. Find the equation of the tangent and normal to the given curves at the given points.

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3) (ii) $y^2 = \dfrac{x^3}{4-x}$ at (2, – 2).

Ans. (i) Tangent : y = 2x + 1, Normal : x + 2y = 7
(ii) Tangent : 2x + y = 2, Normal : x – 2y = 6

3. Prove that area of the triangle formed by any tangent to the curve $xy = c^2$ and coordinate axes is constant.

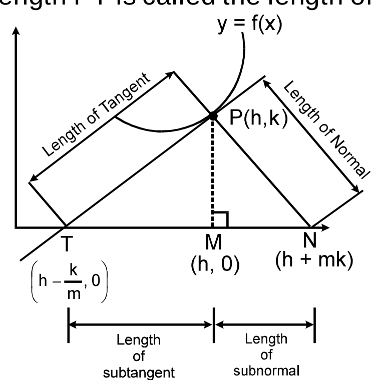
4. How many tangents are possible from origin on the curve $y = (x + 1)^3$. Also find the equation of these tangents. **Ans.** y = 0, 4y = 27x.

5. Find the equation of tangent to the hyperbola $y = \dfrac{x+9}{x+5}$ which passes through (0, 0) origin

Ans. x + y = 0; 25y + x= 0

D. Length of Tangent, Normal

Let P (h, k) be any point on curve y = f(x). Let tangent drawn at point P meets x-axis at T & normal at point P meets x-axis at N. Then the length PT is called the length of tangent and PN is called length of normal.



Projection of segment PT on x-axis, TM, is called the subtangent and similarly projection of line segment PN on x axis is called sub normal.

Let $m = \left.\dfrac{dy}{dx}\right|_{h,k}$ = slope of tangent.

Hence equation of tangent is m (x – h) = (y – k)

putting y = 0 we get x - intercept of tangent $x = h - \dfrac{k}{m}$

similarly the x-intercept of normal is x = h + km

Now, length PT, PN etc can be easily evaluated using distance formula

(i) PT = $\left| k\sqrt{1+\frac{1}{m^2}} \right|$ = Length of Tangent (ii) PN = $\left| k\sqrt{1+m^2} \right|$ = Length of Normal

(iii) TM = $\left| \frac{k}{m} \right|$ = Length of subtangent (iv) MN = |km| = Length of subnormal

Example: Find the length of tangent for the curve $y = x^3 + 3x^2 + 4x - 1$ at point $x = 0$.

Solution. Here $m = \left.\frac{dy}{dx}\right|_{x=0}$ & $k = y(0) \Rightarrow k = -1$

$$\frac{dy}{dx} = 3x^2 + 6x + 4 \Rightarrow m = 4$$

$$\ell = \left| k\sqrt{1+\frac{1}{m^2}} \right| \Rightarrow \ell = \left| -1\sqrt{1+\frac{1}{16}} \right| = \frac{\sqrt{17}}{4}$$

Example: Prove that for the curve $y = be^{x/a}$, the length of subtangent at any point is always constant.

Solution $y = be^{x/a}$ Let the point be (x_1, y_1)

$$\Rightarrow m = \left.\frac{dy}{dx}\right|_{x_1} = \frac{b.e^{x_1/a}}{a} = \frac{y_1}{a}$$

Now, length of subtangent = $\frac{y_1}{m} = \frac{y_1}{y_1/a} = a$ Hence proved.

Example : For the curve $y = a\,\ell n\,(x^2 - a^2)$ show that sum of lengths of tangent & subtangent at any point is proportional to coordinates of point of tangency.

Solution. Let point of tangency be (x_1, y_1)

$$m = \left.\frac{dy}{dx}\right|_{x_1} = \frac{2ax_1}{x_1^2 - a^2}$$

$$\text{tangent + subtangent} = y_1\sqrt{1+\frac{1}{m^2}} + \frac{y_1}{m}$$

$$= y_1\sqrt{1+\frac{(x_1^2-a^2)^2}{4a^2x_1^2}} + \frac{y_1(x_1^2-a^2)}{2ax_1}$$

$$= y_1\frac{\sqrt{x_1^4+a^4+2a^2x_1^2}}{2ax_1} + \frac{y_1(x_1^2-a^2)}{2ax_1}$$

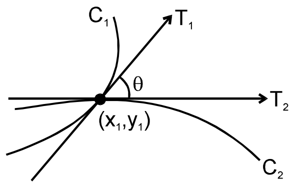
$$= \frac{y_1(x_1^2+a^2)}{2ax_1} + \frac{y_1(x_1^2-a^2)}{2ax_1}$$

$$= \frac{y_1(x_1^2)}{2ax_1} = \frac{x_1\,y_1}{2a}$$

Hence proved.

E Angle between the curves

Angle between two intersecting curves is defined as the acute angle between their tangents or the normals at the point of intersection of two curves.



$$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1m_2} \right|$$

where m_1 & m_2 are the slopes of tangents at the intersection point (x_1, y_1). Note carefully that

(i) The curves must intersect for the angle between them to be defined. This can be ensured by finding their point of intersection or graphically.

(ii) If the curves intersect at more than one point then angle between curves is written with references to the point of intersection.

(iii) Two curves are said to be orthogonal if angle between them at **each** point of intersection is right angle. i.e. $m_1 m_2 = -1$.

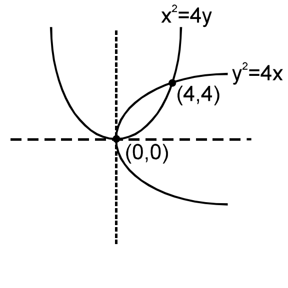
Example : Find angle between $y^2 = 4x$ and $x^2 = 4y$. Are these two curves orthogonal?

Solution. $y^2 = 4x$ and $x^2 = 4y$ intersect at point (0, 0) and (4, 4)

$C_1 : y^2 = 4x$ $C_2 : x^2 = 4y$

$\frac{dy}{dx} = \frac{2}{y}$ $\frac{dy}{dx} = \frac{x}{2}$

$\left.\frac{dy}{dx}\right|_{0,0} = \infty$ $\left.\frac{dy}{dx}\right|_{0,0} = 0$



Hence $\tan\theta = 90°$ at point (0, 0)

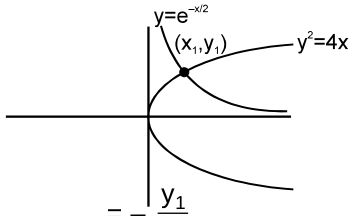
$\left.\dfrac{dy}{dx}\right|_{(4,\,4)} = \dfrac{1}{2}$ $\qquad\left.\dfrac{dy}{dx}\right|_{(4,\,4)} = 2$

$$\tan\theta = \left|\dfrac{2-\dfrac{1}{2}}{1+2.\dfrac{1}{2}}\right| = \dfrac{3}{4}$$

Two curves are not orthogonal because angle at (4, 4) is not 90°.

Example: Find the angle between curves $y^2 = 4x$ and $y = e^{-x/2}$

Solution. Let the curves intersect at point (x_1, y_1)



for $y^2 = 4x$ $\qquad \left.\dfrac{dy}{dx}\right|_{(x_1,\,y_1)} = \dfrac{2}{y_1}$

and for $y = e^{-x/2}$ $\qquad \left.\dfrac{dy}{dx}\right|_{(x_1,\,y_1)} = -\dfrac{1}{2}\,e^{-x_1/2} \qquad = -\dfrac{y_1}{2}$

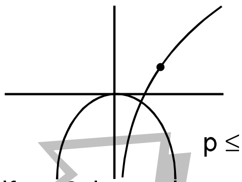
$\Rightarrow \quad m_1 m_2 = -1 \qquad$ Hence $\theta = 90°$

Note : here that we have not actually found the intersection point but geometrically we can see that the curves intersect.

Example : Find possible values of p such that the equation $px^2 = \ell nx$ has exactly one solution.

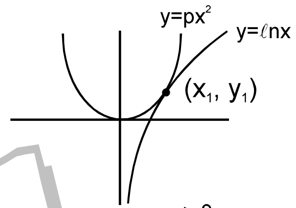
Solution. Two curves must intersect at only one point. Hence

(i)



$p \le 0$

(ii)



$p > 0$

I. if $p \le 0$ then only one solution (see graph)

II. if $p > 0$

then the two curves must only touch each other

i.e. tangent at $y = px^2$ and $y = \ell nx$ must have same slope at point (x_1, y_1)

$\Rightarrow \quad 2px_1 = \dfrac{1}{x_1}$

$\Rightarrow \quad x_1^{\,2} = \dfrac{1}{2p}$ ....(i)

also $\quad y_1 = px_1^{\,2} \quad \Rightarrow \quad y_1 = p\left(\dfrac{1}{2p}\right)$

$\Rightarrow \quad y_1 = \dfrac{1}{2}$ ....(ii)

and $\quad y_1 = \ell nx_1 \quad \Rightarrow \quad \dfrac{1}{2} = \ell nx_1$

$\Rightarrow \quad x_1 = e^{1/2}$ ....(iii)

Hence $\quad x_1^{\,2} = \dfrac{1}{2p} \quad \Rightarrow \quad e = \dfrac{1}{2p} \quad \Rightarrow \quad p = \dfrac{1}{2e}$

Hence possible values of p are $(-\infty, 0] \cup \left\{\dfrac{1}{2e}\right\}$

Self Practice Problems :

1. For the curve $x^{m+n} = a^{m-n}\, y^{2n}$, where a is a positive constant and m, n are positive integers, prove that the m^{th} power of subtangent varies as n^{th} power of subnormal.

2. Prove that the segment of the tangent to the curve $y = \dfrac{a}{2}\ \ell n\dfrac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}} - \sqrt{a^2-x^2}$ contained between the y-axis & the point of tangency has a constant length .

3. A curve is given by the equations $x = at^2$ & $y = at^3$. A variable pair of perpendicular lines through the origin 'O' meet the curve at P & Q . Show that the locus of the point of intersection of the tangents at P & Q is $4y^2 = 3ax - a^2$.

4. Find the length of the subnormal to the curve $y^2 = x^3$ at the point (4, 8). **Ans.** 24

5. Find the angle of intersection of the following curves:

(i) $y = x^2$ & $6y = 7 - x^3$ at (1, 1) $\qquad$ (ii) $x^2 - y^2 = 5$ & $\dfrac{x^2}{18}+\dfrac{y^2}{8} = 1$.

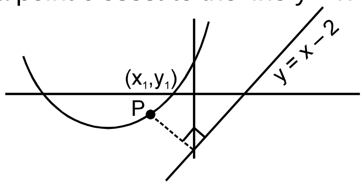
Ans. (i) $\pi/2$ $\qquad$ (ii) $\pi/2$

F. Shortest distance between two curves

Shortest distance between two non-intersecting curves always along the common normal.
(Wherever defined)

Example: Find the shortest distance between the line $y = x - 2$ and the parabola $y = x^2 + 3x + 2$.

Solution. Let $P(x_1, y_1)$ be a point closest to the line $y = x - 2$



then $\left.\dfrac{dy}{dx}\right|_{(x_1, y_1)}$ = slope of line

$\Rightarrow \quad 2x_1 + 3 = 1 \quad \Rightarrow \quad x_1 = -1 \quad \Rightarrow \quad y_1 = 0$

Hence point $(-1, 0)$ is the closest and its perpendicular distance from the line $y = x - 2$ will give the shortest distance

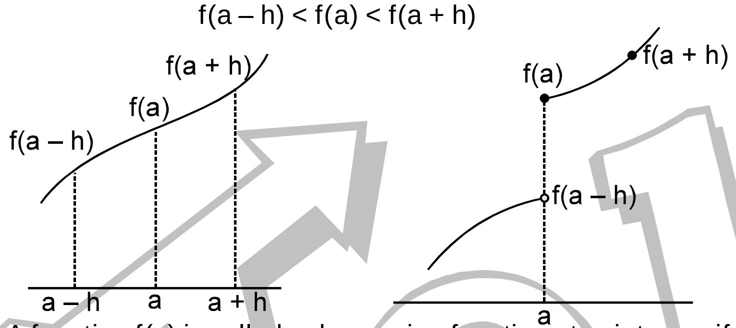
$\Rightarrow \quad p = \dfrac{3}{\sqrt{2}}$.

Monotonocity

A. Monotonocity about a point

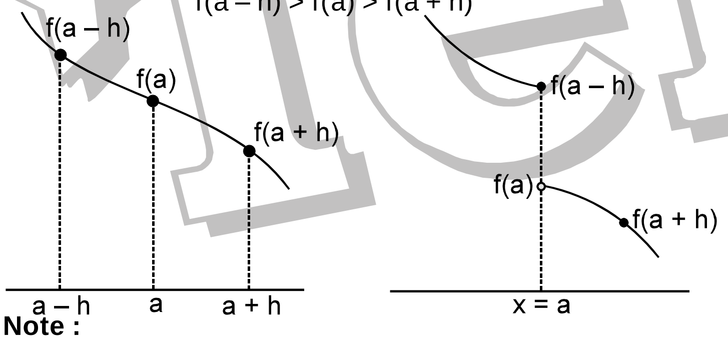
1. A function f(x) is called an increasing function at point x = a. If in a sufficiently small neighbourhood around x = a.

$$f(a - h) < f(a) < f(a + h)$$

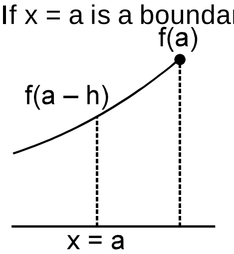


2. A function f(x) is called a decreasing function at point x = a if in a sufficiently small neighbourhood around x = a.

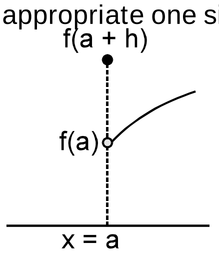
$$f(a - h) > f(a) > f(a + h)$$



Note :
If x = a is a boundary point then use the appropriate one sided inequality to test monotonocity of f(x).



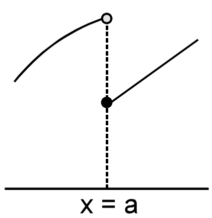
$f(a) > f(a - h)$
increasing at x = a



$f(a + h) < f(a)$
decreasing at x = a

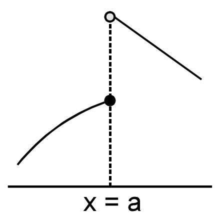
Example : Which of the following functions is increasing, decreasing or neither increasing nor decreasing at x = a.

(i)



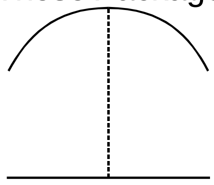
neither increasing nor decreasing

(ii)



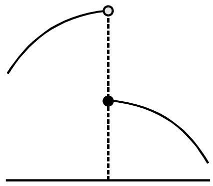
Increasing

(iii)



x = a
neither increasing
nor decreasing

(iv)



x = a
decreasing

3. **Test for increasing and decreasing functions at a point**

(i) If $f'(a) > 0$ then $f(x)$ is increasing at x = a.

(ii) If $f'(a) < 0$ then $f(x)$ is decreasing at x = a.

(iii) If $f'(a) = 0$ then examine the sign of $f'(a^+)$ and $f'(a^-)$.

(a) If $f'(a^+) > 0$ and $f'(a^-) > 0$ then increasing

(b) If $f'(a^+) < 0$ and $f'(a^-) < 0$ then decreasing

(c) otherwise neither increasing nor decreasing.

Example : Let $f(x) = x^3 - 3x + 2$. Examine the nature of function at points x = 0, 1, 2.

Solution : $f(x) = x^3 - 3x + 2$

$f'(x) = 3(x^2 - 1)$

(i) $f'(0) = -3 \quad \Rightarrow \quad$ decreasing at x = 0

(ii) $f'(1) = 0$

also, $f'(1^+)$ = positive and $f'(1^-)$ = negative

$\Rightarrow$ neither increasing nor decreasing at x = 1.

(iii) $f'(2) = 9 \quad \Rightarrow \quad$ increasing at x = 2

Note : Above rule is applicable only for functions that are differentiable at x = a.

B. Monotonocity over an interval

1. A function $f(x)$ is said to be monotonically increasing for all such interval (a, b) where $f'(x) \geq 0$ and equality may hold only for discreet values of x. i.e. $f'(x)$ does not identically become zero for $x \in (a, b)$ or any sub interval.

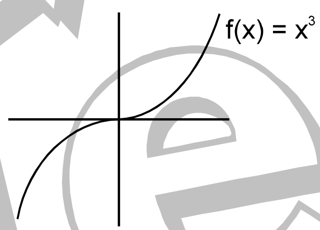
2. $f(x)$ is said to be monotonically decreasing for all such interval (a, b) where $f'(x) \leq 0$ and equality may hold only for discrete values of x.

Note : By discrete, points, we mean that points where $f'(x) = 0$ don't form an interval

For example. Let $f(x) = x^3$

$f'(x) = 3x^2$

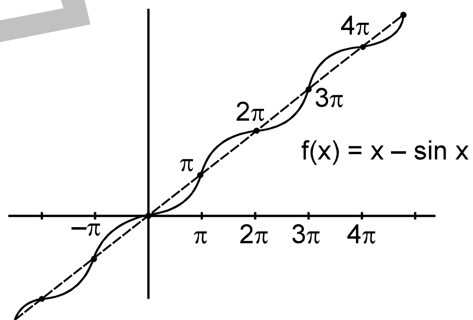
$f'(x) > 0$ every where except at x = 0. Hence $f(x)$ will be considered monotonically increasing function for $x \in R$. also,



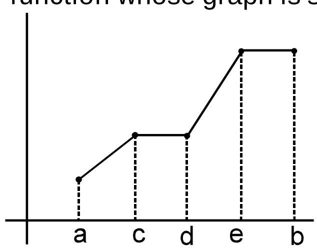
Let $f(x) = x - \sin x$

$f'(x) = 1 - \cos x$

Now, $f'(x) > 0$ every where except at $x = 0, \pm 2\pi, \pm 4\pi$ etc. but all these points are discrete and donot form an interval hence we can conclude that $f(x)$ is monotonically increasing for $x \in R$. In fact we can also see it graphically.



Let us consider another function whose graph is shown for $x \in (a, b)$.



Here also $f'(x) \geq 0$ for all $x \in (a, b)$ but note that in this case equality of $f'(x) = 0$ holds for all $x \in (c, d)$ and (e, b). Here $f'(x)$ become identically zero and hence the given function cannot be assumed to be monotonically increasing for $x \in (a, b)$.

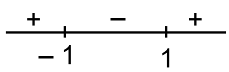
Example : Find the interval where $f(x) = x^3 - 3x + 2$ is monotonically increasing.

Solution. $f'(x) = x^3 - 3x + 2$

$f'(x) = 3(x^2 - 1)$

$f'(x) = 3(x - 1)(x + 1)$

for M.I. $f'(x) \geq 0 \quad \Rightarrow \quad 3(x - 1)(x + 1) \geq 0$



$\Rightarrow \quad x \in [-\infty, -1] \cup [1, \infty)$

Note :

(i) A function is said to be monotonic if it's either increasing or decreasing.

(ii) The points for which $f'(x)$ is equal to zero or doesn't exist are called **critical points**. Here it should also be noted that critical points are the interior points of an interval.

(iii) The stationary points are the points where $f'(x) = 0$ in the domain.

Example : Find the intervals of monotonicity of following functions.

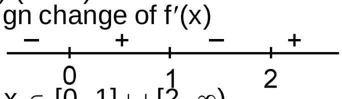
(i) $f(x) = x^2 (x-2)^2$ (ii) $f(x) = x \,\ell n\, x$

(iii) $f(x) = \sin x + \cos x \quad ; \quad x \in [0, 2\pi]$

Solution. (i) $f(x) = x^2 (x-2)^2$

$f'(x) = 4x (x-1)(x-2)$

observing the sign change of $f'(x)$



Hence M.I. for $x \in [0, 1] \cup [2, \infty)$

and M.D. for $x \in (-\infty, 0] \cup [1, 2]$

Note : Closed bracket can be used for both M.I. as well as M.D. In above example $x = 1$ is boundary point for $x \in [0, 1]$ and since $f(1) > f(1-h)$. So we can say $f(x)$ is M.I. at $x = 1$ for $x \in [0, 1]$. However also note that for the interval $x \in [1, 2]$ again $x = 1$ becomes a boundary point and $f(1) > f(1+h)$. Hence $f(x)$ is M.D. at $x = 1$ for $x \in [1, 2]$

(ii) $f(x) = x \,\ell n\, x$

$f'(x) = 1 + \ell n\, x$

$f'(x) \geq 0 \quad \Rightarrow \quad \ell n\, x \geq -1 \quad \Rightarrow \quad x \geq \dfrac{1}{e}$

$\Rightarrow$ M.I. for $x \in \left[\dfrac{1}{e}, \infty\right)$ and M.D for $x \in \left(0, \dfrac{1}{e}\right]$.

(iii) $f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

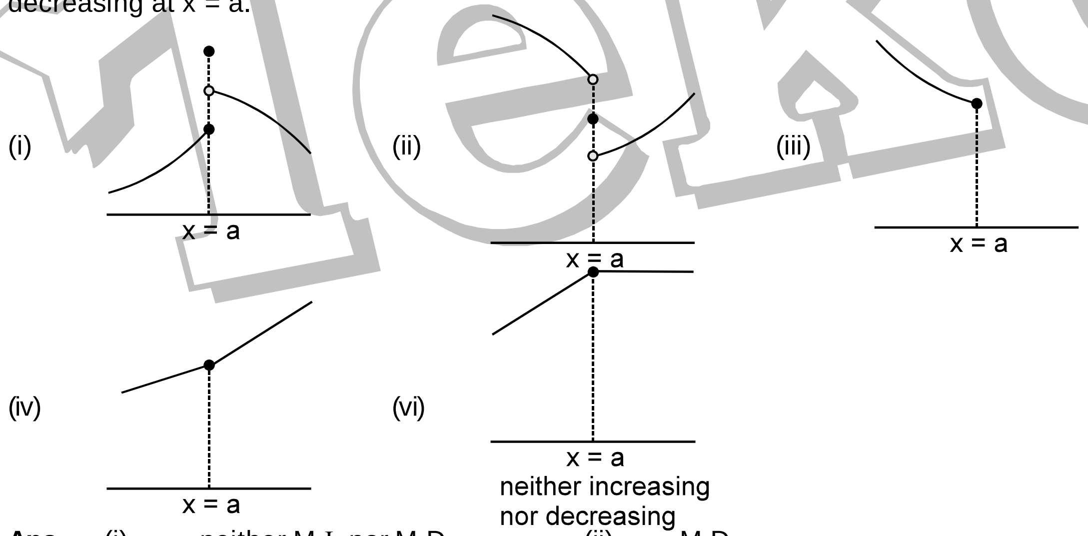
for M.I. $f'(x) \geq 0 \quad \Rightarrow \quad \cos x \geq \sin x$

$\Rightarrow \quad x \in \left[0, \dfrac{\pi}{4}\right] \cup \left[\dfrac{5\pi}{4}, 2\pi\right]$

therefore M.D. for $x \in \left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

Exercise

1. For each of the following graph comment whether f(x) is increasing or decreasing or neither increasing nor decreasing at x = a.



Ans. (i) neither M.I. nor M.D. (ii) M.D.
(iii) M.D (iv) M.I.

2. Let $f(x) = x^3 - 3x^2 + 3x + 4$, comment on the monotonic behaviour of f(x) at (i) x = 0 (ii) x = 1.

Ans. M.I. both at x = 0 and x = 1.

3. Draw the graph of function $f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ [x] & 1 \leq x \leq 2 \end{cases}$. Graphically comment on the monotonic behaviour of f(x) at x = 0, 1, 2. Is f(x) M.I. for $x \in [0, 2]$?

Ans. M.I. at x = 0, 2 ; neither M.I. nor M.D. at x = 1. No, f(x) is not M.I. for $x \in [0, 2]$.

4. Find the intervals of monotonicity of the following functions.

(i) $f(x) = -x^3 + 6x^2 - 9x - 2$ **Ans.** I in $[1, 3]$; D in $(-\infty, 1] \cup (3, \infty)$

(ii) $f(x) = x + \dfrac{1}{x+1}$ **Ans.** I in $(-\infty, -2] \cup [0, \infty)$; D in $[-2, -1) \cup (-1, 0]$

(iii) $f(x) = x \,.\, e^{x - x^2}$ **Ans.** I in $\left[-\dfrac{1}{2}, 1\right]$; D in $\left(-\infty, -\dfrac{1}{2}\right] \cup [1, \infty)$

(iv) $f(x) = x - \cos x$ **Ans.** I for $x \in R$

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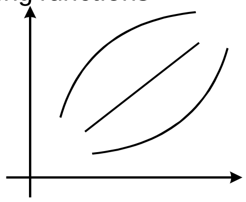
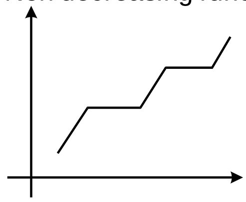
C. Classification of functions

Depending on the monotonic behaviour, functions can be classified into following cases.

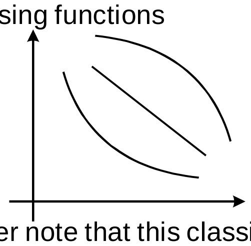
1. Increasing functions



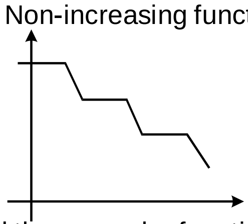
2. Non decreasing functions



3. Decreasing functions



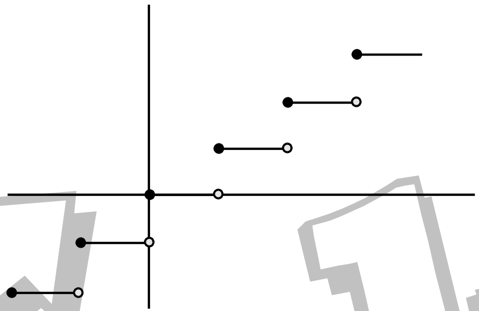
4. Non-increasing functions



However note that this classification is not complete and there may be function which cannot be classified into any of the above cases for some interval (a, b).

Example : $f(x) = [x]$ is a step up function. Is it a monotonically increasing function for $x \in R$.

Solution. No, $f(x) = [x]$ is not M.I. for $x \in R$ rather, it is a non-decreasing function as illustrated by its graph.



Example : If $f(x) = \sin^4x + \cos^4x + bx + c$, then find possible values of b and c such that f(x) is monotonic for all $x \in R$

Solution. $f(x) = \sin^4x + \cos^4x + bx + c$

$f'(x) = 4\sin^3x\cos x - 4\cos^3x\sin x + b = -\sin 4x + b$.

(i) for M.I. $f'(x) \ge 0$ for all $x \in R$

$b \ge \sin 4x$ for all $x \in R \Rightarrow b \ge 1$

(ii) for M.D. $f'(x) \le 0$ for all $x \in R$

$b \le \sin 4x$ for all $x \in R \Rightarrow b \le -1$

Hence for f(x) to be monotonic $b \in (-\infty, -1] \cup (1, \infty)$ and $c \in R$.

Example : Find possible values of a such that $f(x) = e^{2x} - (a + 1)e^x + 2x$ is monotonically increasing for $x \in R$

Solution. $f(x) = e^{2x} - (a + 1)e^x + 2x$

$f'(x) = 2e^{2x} - (a + 1)e^x + 2$

Now, $2e^{2x} - (a + 1)e^x + 2 \ge 0$ for all $x \in R$

$\Rightarrow 2\left(e^x + \frac{1}{e^x}\right) - (a + 1) \ge 0$ for all $x \in R$

$(a + 1) \le 2\left(e^x + \frac{1}{e^x}\right)$ for all $x \in R$

$\Rightarrow a + 1 \le 4 \quad \left(\because e^x + \frac{1}{e^x} \text{ has minimum value } 2\right) \Rightarrow a \le 3$

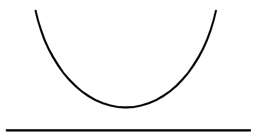
Aliter

$2e^{2x} - (a + 1)e^x + 2 \ge 0$ for all $x \in R$

putting $e^x = t$; $t \in (0, \infty)$

$2t^2 - (a + 1)t + 2 \ge 0$ for all $t \in (0, \infty)$

Hence either



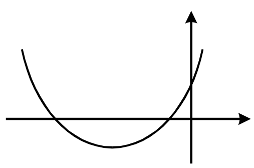
(i) $D \le 0$

$\Rightarrow (a + 1)^2 - 4 \le 0$

$\Rightarrow (a + 5)(a - 3) \le 0$

$\Rightarrow a \in [-5, 3]$

or



(ii) both roots are negative

$D \ge 0 \quad \& \quad -\frac{b}{2a} < 0 \quad \& \quad f(0) \ge 0$

$\Rightarrow a \in (-\infty, -5] \cup [3, \infty) \quad \& \quad \frac{a+1}{4} < 0 \quad \& \quad 2 \ge 0$

$\Rightarrow a \in (-\infty, -5] \cup [3, \infty) \quad \& \quad a < -1 \quad \& \quad a \in R$

$\Rightarrow a \in (-\infty, -5]$

Taking union of (i) and (ii), we get $a \in (-\infty, 3]$.

Exercise

1. Let $f(x) = x - \tan^{-1}x$. Prove that f(x) is monotonically increasing for $x \in R$.

2. If $f(x) = 2e^x - ae^{-x} + (2a + 1)x - 3$ monotonically increases for $\forall x \in R$, then find range of values of a

Ans. $a \geq 0$

3. Let $f(x) = e^{2x} - ae^x + 1$. Prove that f(x) cannot be monotonically decreasing for $\forall x \in R$ for any value of 'a'.

4. Find range of values of 'a' such that $f(x) = \sin 2x - 8(a + 1)\sin x + (40 - 10)x$ is monotonically decreasing $\forall x \in R$ **Ans.** $a \in [-4, 0]$

5. If $f(x) = x^3 + (a + 2)x^2 + 5ax + 5$ is a one-one function then find values of a. **Ans.** $a \in [1, 4]$

D. Proving Inequalities

Comparision of two functions f(x) and g(x) can be done by analysing their monotonic behavior or graph.

Example : For $x \in \left(0, \frac{\pi}{2}\right)$ prove that $\sin x < x < \tan x$

Solution. Let $f(x) = x - \sin x \Rightarrow f'(x) = 1 - \cos x$

$f'(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow$ f(x) is M.I. $\Rightarrow f(x) > f(0)$

$\Rightarrow x - \sin x > 0 \Rightarrow x > \sin x$

Similarly consider another function $g(x) = x - \tan x \Rightarrow g'(x) = 1 - \sec^2x$

$g'(x) < 0$ for $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow$ g(x) is M.D.

Hence $g(x) < g(0)$

$x - \tan x < 0 \Rightarrow x < \tan x$

$\sin x < x < \tan x$ **Hence proved**

Example : For $x \in (0, 1)$ prove that $x - \frac{x^3}{3} < \tan^{-1}x < x - \frac{x^3}{6}$ hence or otherwise find $\lim_{x\to 0}\left[\frac{\tan^{-1}x}{x}\right]$

Solution. Let $f(x) = x - \frac{x^3}{3} - \tan^{-1}x$

$f(x) = 1 - x^2 - \frac{1}{1+x^2}$

$f'(x) = -\frac{x^4}{1+x^2}$

$f'(x) < 0$ for $x \in (0, 1) \Rightarrow$ f(x) is M.D.

$\Rightarrow f(x) < f(0)$

$\Rightarrow x - \frac{x^3}{3} - \tan^{-1}x < 0$

$\Rightarrow x - \frac{x^3}{3} < \tan^{-1}x$ ....(i)

Similarly $g(x) = x - \frac{x^3}{6} - \tan^{-1}x$

$g'(x) = 1 - \frac{x^2}{2} - \frac{1}{1+x^2}$

$g'(x) = \frac{x^2(1-x^2)}{2(1+x^2)}$

$g'(x) > 0$ for $x \in (0, 1) \Rightarrow$ g(x) is M.I.

$\Rightarrow g(x) > g(0)$

$x - \frac{x^3}{6} - \tan^{-1}x > 0$

$x - \frac{x^3}{6} > \tan^{-1}x$ ....(ii)

from (i) and (ii), we get

$x - \frac{x^3}{3} < \tan^{-1}x < x - \frac{x^3}{6}$ **Hence Proved**

Also, $1 - \frac{x^2}{3} < \frac{\tan^{-1}x}{x} < 1 - \frac{x^2}{6}$

Hence by sandwich theorem we can prove that $\lim_{x\to 0}\frac{\tan^{-1}x}{x} = 1$ but it must also be noted that as $x \to 0$, value of $\frac{\tan^{-1}x}{x} \longrightarrow 1$ from left hand side i.e. $\frac{\tan^{-1}x}{x} < 1$

$\Rightarrow \lim_{x\to 0}\left[\frac{\tan^{1}x}{x}\right] = 0$

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NOTE : In proving inequalities, we must always check when does the equality takes place because the point of equality is very important in this method. Normally point of equality will occur at the end point of intervals or will be easy to be predicated by hit and trial

Example : For $x \in \left(0, \frac{\pi}{2}\right)$, prove that $\sin x > x - \frac{x^3}{6}$

Solution Let $f(x) = \sin x - x + \frac{x^3}{6}$

$f'(x) = \cos x - 1 + \frac{x^2}{2}$

we cannot decide at this point wether $f'(x)$ is positive or negative, hence let us check for monotonic nature of $f'(x)$

$f''(x) = x - \sin x$

Since $f''(x) > 0 \Rightarrow f'(x)$ is M.I. for $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow f'(x) > f(0)$

$\Rightarrow f'(x) > 0 \Rightarrow f(x)$ is M.I.

$\Rightarrow f(x) > f(0)$

$\Rightarrow \sin x - x + \frac{x^3}{6} > 0 \Rightarrow \sin x > x - \frac{x^3}{6}$ **Hence proved**

Example : Examine which is greater $\sin x \tan x$ or x^2. Hence evaluate $\lim\limits_{x\to 0}\left[\frac{\sin x \tan x}{x^2}\right]$, where $x \in \left(0, \frac{\pi}{2}\right)$

Solution Let $f(x) = \sin x . \tan x - x^2$

$f'(x) = \cos x . \tan x + \sin x . \sec^2 x - 2x$

$\Rightarrow f'(x) = \sin x + \sin x \sec^2 x - 2x$

$\Rightarrow f''(x) = \cos x + \cos x \sec^2 x + 2\sec^2 x \sin x \tan x - 2$

$\Rightarrow f''(x) = (\cos x + \sec x - 2) + 2\sec^2 x \sin x \tan x$

Now $\cos x + \sec x - 2 = \left(\sqrt{\cos x} - \sqrt{\sec x}\right)^2$ and $2\sec^2 x \tan x . \sin x > 0$ because $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow f''(x) > 0 \Rightarrow f'(x)$ is M.I.

Hence $f'(x) > f'(0)$

$\Rightarrow f'(x) > 0 \Rightarrow f(x)$ is M.I. $\Rightarrow f(x) > 0$

$\Rightarrow \sin x \tan x - x^2 > 0$

Hence $\sin x \tan x > x^2$

$\Rightarrow \frac{\sin x \tan x}{x^2} > 1 \Rightarrow \lim\limits_{x\to 0}\left[\frac{\sin x \tan x}{x^2}\right] = 1$

Example : Prove that $f(x) = \left(1+\frac{1}{x}\right)^x$ is monotonically increasing in its domain. Hence or otherwise draw graph of f(x) and find its range

Solution. $f(x) = \left(1+\frac{1}{x}\right)^x$, for Domain of f(x) $1 + \frac{1}{x} > 0$

$\Rightarrow \frac{x+1}{x} > 0 \Rightarrow (-\infty, -1) \cup (0, \infty)$

Consider $f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) + \frac{x}{1+\frac{1}{x}} \frac{-1}{x^2}\right]$

$\Rightarrow f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right]$

Now $\left(1+\frac{1}{x}\right)^x$ is always positive, hence the sign of $f'(x)$ depends on sign of $\ell n\left(1+\frac{1}{x}\right) - \frac{1}{1+x}$

i.e. we have to compare $\ell n\left(1+\frac{1}{x}\right)$ and $\frac{1}{1+x}$

So lets assume $g(x) = \ell n\left(1+\frac{1}{x}\right) - \frac{1}{x+1}$

$g'(x) = \frac{1}{1+\frac{1}{x}} \frac{-1}{x^2} + \frac{1}{(x+1)^2} \Rightarrow g'(x) = \frac{-1}{x(x+1)^2}$

(i) for $x \in (0, \infty)$, $g'(x) < 0$

$\Rightarrow g(x)$ is M.D. for $x \in (0, \infty)$

$g(x) > \lim\limits_{x\to\infty} g(x)$

$g(x) > 0$.

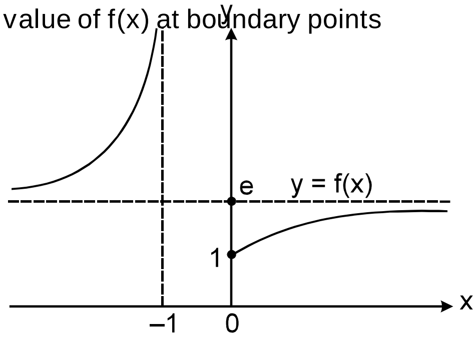
and since $g(x) > 0 \Rightarrow f'(x) > 0$
(ii) for $x \in (-\infty, -1)$, $g'(x) > 0$

$\Rightarrow$ $g(x)$ is M.I. for $x \in (-\infty, -1)$ $\Rightarrow$ $g(x) > \lim\limits_{x\to-\infty} g(x)$
$\Rightarrow$ $g(x) > 0$ $\Rightarrow$ $f'(x) > 0$
Hence from (i) and (ii) we get $f'(x) > 0$ for all $x \in (-\infty, -1) \cup (0, \infty)$
$\Rightarrow$ $f(x)$ is M.I. in its Domain
For drawing the graph of f(x), its important to find the value of f(x) at boundary points
i.e. $\pm\infty$, 0, –1

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e$$

$$\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x = 1 \quad \text{and} \quad \lim_{x\to -1}\left(1+\frac{1}{x}\right)^x = \infty$$

so the graph of f(x) is



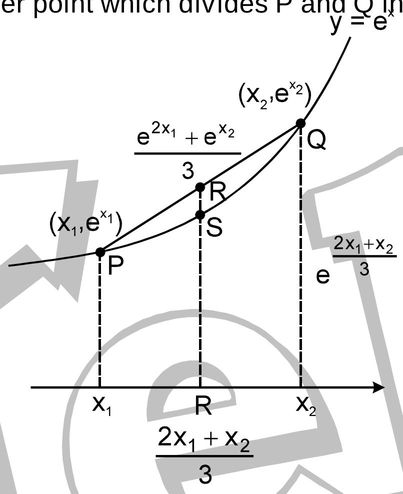
Range is $y \in (1, \infty) - \{e\}$

E. Proving inequalities using graph

Generally these inequalities involve comparison between values of two functions at some particular points.

Example : Prove that for any two numbers x_1 & x_2, $\dfrac{e^{2x_1}+e^{x_2}}{3} > e^{\frac{2x_1+x_2}{3}}$

Solution. Assume $f(x) = e^x$ and let x_1 & x_2 be two points on the curve $y = e^x$.
Let R be another point which divides P and Q in ratio 1 : 2.



y coordinate of point R is $\dfrac{e^{2x_1}+e^{x_2}}{3}$ and y coordinate of point S is $e^{\frac{2x_1+x_2}{3}}$. Since $f(x) = e^x$ is always concave up, hence point R will always be above point S.

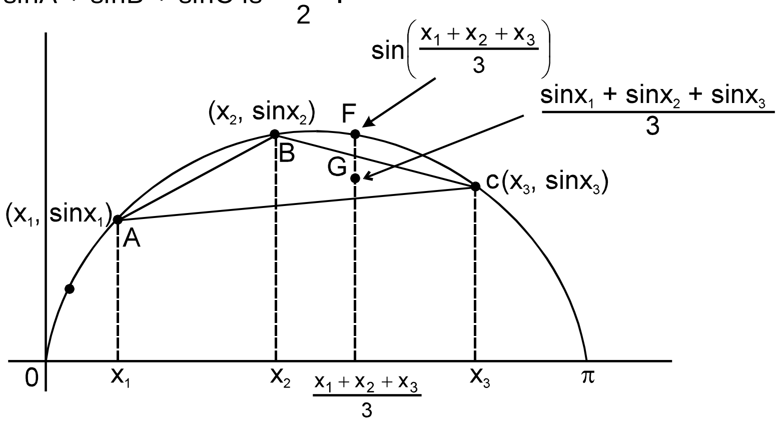
$$\Rightarrow \frac{e^{2x_1}+e^{x_2}}{3} < e^{\frac{2x_1+x_2}{3}}$$

(above inequality could also be easily proved using AM and GM.)

Example : If $0 < x_1 < x_2 < x_3 < \pi$ then prove that $\sin\left(\dfrac{x_1+x_2+x_3}{3}\right) > \dfrac{\sin x_1+\sin x_2+\sin x_3}{3}$. Hence or otherwise prove that if A, B, C are angles of a triangle then maximum value of

sinA + sinB + sinC is $\dfrac{3\sqrt{3}}{2}$.

Solution.



Let point A, B, C form a triangle y coordinate of centroid G is $\dfrac{\sin x_1+\sin x_2+\sin x_3}{3}$ and y coordinate of point F is $\sin\left(\dfrac{x_1+x_2+x_3}{3}\right)$.

Hence $\sin\left(\frac{x_1+x_2+x_3}{3}\right) > \frac{\sin x_1+\sin x_2+\sin x_3}{3}$.

if $A + B + C = \pi$, then

$$\sin\left(\frac{A+B+C}{3}\right) > \frac{\sin A+\sin B+\sin C}{3} \quad\Rightarrow\quad \sin\frac{\pi}{3} > \frac{\sin A+\sin B+\sin C}{3}$$

$\Rightarrow \quad \frac{3\sqrt{3}}{2} > \sin A + \sin B + \sin C$

$\Rightarrow \quad$ maximum value of $(\sin A + \sin B + \sin C) = \frac{3\sqrt{3}}{2}$

Example : Compare which of the two is greater $(100)^{1/100}$ or $(101)^{1/101}$.

Solution. Assume $f(x) = x^{1/x}$ and let us examine monotonic nature of f(x)

$f'(x) = x^{1/x} \cdot \left(\frac{1-\ell nx}{x^2}\right)$

$f'(x) > 0 \quad\Rightarrow\quad x \in (0,e)$

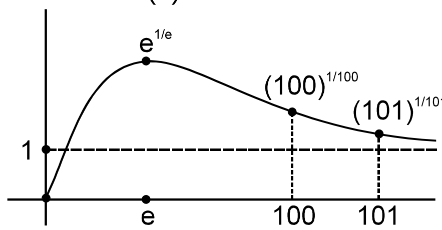
and $f'(x) < 0 \quad\Rightarrow\quad x \in (e,\infty)$

Hence f(x) is M.D. for $x \ge e$

and since $100 < 101$

$\Rightarrow \quad f(100) > f(101)$

$\Rightarrow \quad (100)^{1/100} > (101)^{1/101}$



Exercise

1. Prove the following inequalities

(i) $x < -\ell n(1 - x)$ for $x \in (0, 1)$

(ii) $x > \tan^{-1}(x)$ for $x \in (0, \infty)$

(iii) $e^x > x + 1$ for $x \in (0, \infty)$

(iv) $\frac{x}{1+x} \le \ell n\,(1 + x) \le x$ for $x \in (0, \infty)$

(v) $\frac{2}{\pi} < \frac{\sin x}{x} < 1$ for $x \in \left(0, \frac{\pi}{2}\right)$

2. Identify which is greater $\frac{1+e^2}{e}$ or $\frac{1+\pi^2}{\pi}$ **Ans.** $\frac{1+e^2}{e}$

3. If $0 < x_1 < x_2 < x_3 < \pi$, then prove that

$$\sin\left(\frac{2x_1+x_2+x_3}{4}\right) > \frac{2\sin x_1+\sin x_2+\sin x_3}{4}$$

4. If f(x) is monotonically decreasing function and $f''(x) > 0$. Assuming $f^{-1}(x)$ exists prove that

$$\frac{f^{-1}(x_1)+f^{-1}(x_2)}{2} > f^{-1}\left(\frac{x_1+x_2}{2}\right).$$

5. Using $f(x) = x^{1/x}$, identify which is larger e^π or π^e. **Ans.** e^π

F. Mean Value of Theorems

(a) Rolle's Theorem:

Let f(x) be a function of x subject to the following conditions:

(i) f(x) is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) = f(b)$.

Then there exists at least one point x = c such that $f'(c) = 0 \;\forall\; c \in (a,b)$.

(b) LMVT Theorem:

Let f(x) be a function of x subject to the following conditions:

(i) f(x) is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$. **(iii)** $f(a) \ne f(b)$.

Then there exists at least one point x = c such that $a < c < b$ where $f'(c) = \frac{f(b)-f(a)}{b-a}$

Geometrically, the slope of the secant line joining the curve at x = a & x = b is equal to the slope of the tangent line drawn to the curve at x = c. **Note the following:**

* Rolle's theorem is a special case of LMVT since

$$f(a) = f(b) \Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a} = 0.$$

(c) Application Of Rolles Theorem For Isolating The Real Roots Of An Equation f(x) = 0

Suppose a & b are two real numbers such that;

(i) f(x) & its first derivative $f'(x)$ are continuous for $a \le x \le b$.

(ii) f(a) & f(b) have opposite signs.

(iii) $f'(x)$ is different from zero for all values of x between a & b.

Then there is one & only one real root of the equation f(x) = 0 between a & b.

Example : If 2a + 3b + 6c = 0 then prove that the equation $ax^2 + bx + c = 0$ has atleast one real root between 0 and 1.

Solution. Let $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$

$f(0) = 0$

and $\quad f(1) = \frac{a}{3} + \frac{b}{2} + c = 2a + 3b + 6c = 0$

If $\quad f(0) = f(1)$ then $f'(x) = 0$ for some value of $x \in (0, 1)$

$\Rightarrow \quad ax^2 + bx + c = 0$ for atleast one $x \in (0, 1)$

Example : Verify Rolles throrem for $f(x) = (x - a)^n (x - b)^m$ where m, n are natrual numbers for $x \in [a, b]$.

Solution. Being a polynomial function f(x) is continuous as well as differentiable, f(a) = 0 and f(b) = 0

$\Rightarrow \quad f'(x) = 0$ for some $x \in (a , b)$

$n(x - a)^{n-1} (x - b)^m + m(x - a)^n (x - b)^{m-1} = 0$

$\Rightarrow \quad (x - a)^{n-1} (x - b)^{m-1} \; [(m + n)\, x - (nb + ma)] = 0$

$\Rightarrow \quad x = \frac{nb + ma}{m + n}$, which lies in the interval (a, b)

Example : Verify LMVT for $f(x) = - x^2 + 4x - 5$ and $x \in [-1, 1]$

Solution. $f(1) = -2 \quad ; \quad f(-1) = -10$

$\Rightarrow \quad f'(c) = \frac{f(1) - f(-1)}{1 - (-1)}$

$\Rightarrow \quad -2c + 4 = 4 \quad \Rightarrow \quad c = 0$

Example : Using mean value theorem, prove that if b > a > 0, then $\frac{b - a}{1 + b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b - a}{1 + a^2}$

Solution. Let $f(x) = \tan^{-1} x$; $x \in [a, b]$ applying LMVT

$f'(c) = \frac{\tan^{-1} b - \tan^{-1} a}{b - a}$ for a < c < b and $f'(x) = \frac{1}{1 + x^2}$,

Now $f'(x)$ is a monotonically decreasing function

Hence if a < c < b $\quad \Rightarrow \quad f'(b) < f'(c) < f'(a)$

$\Rightarrow \quad \frac{1}{1 + b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b - a} < \frac{1}{1 + a^2}$ **Hence proved**

Maxima - Minima

A. I[st] Fundamental Theorem

1. A function f(x) is said to have a local maximum at x = a if $f(a) > f(x) \ \forall \ x \in (a - h, a + h)$. Where h is a very small positive arbitrary number.

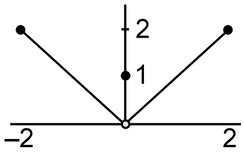
Note : The local maximum of a function is the largest value only in neighbourhood of point x = a.

2. A function f(x) is said to have local minimum at x = a if $f(a) < f(x) \ \forall \ x \in (a - h, a + h)$.

First fundamental theorem is applicable to all functions continuous, discontinuous, differentiable or nondifferentiable at x = a.

Example : Let $f(x) = \begin{cases} |x| & 0 < |x| \le 2 \\ 1 & x = 0 \end{cases}$. Examine the behaviour of f(x) at x = 0.

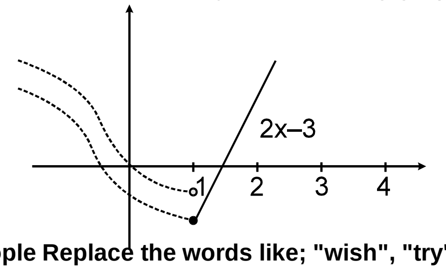
Solution. f(x) has local maxima at x = 0.



Example : Let $f(x) = \begin{cases} -x^3 + \frac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} & 0 \le x < 1 \\ 2x - 3 & 1 \le x \le 3 \end{cases}$

Find all possible values of b such that f(x) has the smallest value at x = 1.

Solution. Such problems can easily solved using graphical approach.



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